ON SIMPLE GROUPS WITH A QUATERNION MAXIMAL 2-SYLOW INTERSECTION

BY

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ABSTRACT

Finite simple groups G with a generalized quaternion maximal 2-Sylow intersection V are determined under the assumption that $[G: N_G(V)]$ is odd.

1. Introduction

This paper is a second step toward classification of simple groups with a maximal 2-Sylow intersection of 2-rank one. The first step was taken in [5], where the cyclic case was considered. We prove the following:

THEOREM 1. *Let G be a simple group and suppose that V is a maximal S_2-intersection of G satisfying the following conditions*:

(i) *V is generalized quaternion,*
(ii) *$|G: N_G(V)|$ is odd, and*
(iii) *$N_G(V)$ is solvable.*

Then an S_2-subgroup of G is either wreathed or quasidihedral.

THEOREM 2. *Conditions* (i) *and* (ii) *of Theorem* 1 *imply condition* (iii).

The proof of Theorem 1 (Section 2) is very simple, since the situation can be readily reduced to that of a simple group with a Sylow 2-subgroup of 2-rank 2, and all such simple groups were recently classified [2].

Our main result is the proof of Theorem 2, which is described in Section 3.

The notation is standard. If H is a group, $K \subseteq H$ ($K \subset H$) means that K is a (proper) subgroup of H. We will say that V is an *S_2-intersection* of G if it is an intersection of two distinct Sylow 2-subgroups of G. If all S_2-intersections of G are trivial, then G is called a *TI-group*. The groups $PSL(2,2^n)$, $n \geq 2$, $PSU(3,2^n)$, $n \geq 2$ and $Sz(2^n)$, $n \geq 3$, will be called *simple Bender-groups*.

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2. Proof of Theorem 1

Let $H = N_G(V)$; then H/V is a non-2-closed, solvable TI-group. Hence by [7], H/V is of 2-rank 1 and it follows by the simplicity of G that G is of 2-rank 2. If $R \in \mathrm{Syl}_2(H)$, then $R \in \mathrm{Syl}_2(G)$ and $V \subset R$ implies that R is not dihedral. Since G is not a TI-group, G is not isomorphic to $PSU(3,4)$. Thus by [2, Prop. 1] and by [6], R is either wreathed or quasidihedral.

3. Proof of Theorem 2

We shall reach a contradiction under the assumption that $H = N_G(V)$ is nonsolvable. Let R denote a 2-Sylow subgroup of H (hence of G) and let S be the maximal solvable normal subgroup of H. The contradiction will be reached in a number of steps.

(a) *V is a normal S_2-subgroup of S.*

PROOF. Let $V_1 \in \mathrm{Syl}_2(S)$; then $V \subseteq V_1$ and $H = SN_H(V_1)$. Since H is nonsolvable, $N_H(V_1)$ is not 2-closed and the maximality of V forces $V_1 = V$.

(b) *There exists a normal subgroup L of H containing S, such that* $[H:L]$ *is odd and L/S is isomorphic to a simple Bender group.*

PROOF. By (a) and the maximality of V, H/S is a nonsolvable TI-group. It follows then by [7, Th. 2] that there exists a normal subgroup L/S of H/S of odd index, which is isomorphic to a simple Bender-group, yielding (b).

For $K \subseteq H$, let $\bar{K}$ denote KS/S. Denote $Z(R)$ by Z, and let z be the involution in V.

(c) *Either* $Z = Z(V)$ *or* $\Omega_1(R) \subseteq Z$.

PROOF. Since $\bar{Z} \lhd N_{\bar{L}}(\bar{R})$, it follows by (b) and the structure of simple Bender groups that either $\bar{Z} = 1$ or $\bar{Z} = \Omega_1(\bar{R})$. Thus either $Z = Z(V)$ or $\Omega_1(R) \subseteq Z$.

(d) $Z = Z(V)$.

PROOF. Suppose that $\Omega_1(R) \subseteq Z$. Then by [3, Cor. 1] there exists an involution y in R and $g \in G$ such that $y = z^g \neq z$. Since $z, y \in Z$, we may choose $g \in N_G(R)$. But then $V^g \lhd R$, $V^g \cap V = 1$ and consequently $\bar{R}$ contains a normal subgroup isomorphic to V, which is not the case in simple Bender groups.

(e) $R = VC_R(V)$.

PROOF. Since H/L and S/V are of odd order and L/S is a simple group it follows that $H/VC_G(V)$ is of odd order. Thus $R \subseteq VC_G(V)$, yielding (e).

(f) *Let $Z(R/V) = R_1/V$. Then $R_1 = VD$, where $V \cap D = Z$ and $D \simeq Q_8$.*

PROOF. By (e), $R_1 = VC_{R_1}(V)$. Let $D = C_{R_1}(V)$; then obviously $R_1 = VD$ and $V \cap D = Z$.

Suppose that u is an involution in $D - Z$. Since $\bar{L}$ is a simple Bender-group, all involutions in $\bar{R}$ are conjugate by $N_{\bar{L}}(\bar{R})$. Thus there exist $g_i \in N_L(R)$, $i = 1, \cdots, r$, such that $Z(\bar{R}) = \Omega_1(\bar{R}) = \{S, uS, u^{g_1}S, \cdots, u^{g_r}S\}$ and

$$R_1/V = Z(R/V) = \Omega_1(R/V) = \{V, uV, u^{g_1}V, \cdots, u^{g_r}V\}.$$

Clearly $V, R_1 \lhd N_L(R)$, hence $D = C_{R_1}(V) \lhd N_L(R)$ and it follows that $u^{g_i} \in D$, $i = 1, \cdots, r$. Thus $D = Z \cup uZ \cup u^{g_1}Z \cup \cdots \cup u^{g_r}Z$ and consequently D is elementary abelian. Since $R_1 = VD$, it follows that $D = \Omega_1(R)$ and by the recent results of Goldschmidt [4], G is either a TI-group or it has an abelian S_2-subgroup, in contradiction to our assumptions.

Thus D contains only one involution and since D/Z is elementary abelian of order at least 4, it follows that $D \simeq Q_8$.

(g) *A contradiction.*

PROOF. Let x be an involution in $C = C_R(V)$; then $x \in R_1$ since $Z(R/V) = \Omega_1(R/V)$ and consequently $x \in D$. It follows by (f) that C contains only one involution, namely the one in V. Thus C is quaternion, and by (e), R is of 2-rank 2. Since R is neither dihedral, nor an S_2-subgroup of a simple TI-group, it follows by [2, Prop. 1] and by [6] that R is either wreathed or quasidihedral. The first case is impossible, since by (d), $Z(R)$ is of order 2. Finally, R cannot be quasidihedral, since then by [1, Lem 1] we would have $[N_R(C): C] = 2$, which is not the case by (e).

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